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Analysis of the spherical Raman–Nath equation

P Bosco†, J Gallardo† and G Dattoli‡

† Quantum Institute, University of California at Santa Barbara, Santa Barbara, California 93106, USA

‡ Comitato Nazionale per la Ricerca e per lo Sviluppo dell'Energia Nucleare e delle Energie Alternative (ENEA), Centro Ricerche Energia, Frascati, CP 65-00044 Frascati, Italy

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Abstract. In this paper we discuss the solution of the spherical Raman–Nath differential equation. This type of equation appears in diverse physical problems, one of which is stimulated Compton scattering. The solution technique exploits the generalisation of an operatorial approach successfully applied to differential-difference equations which are particular cases of the one discussed here.

It has been shown (Dattoli and Renieri 1984) that the analysis of the stimulated Compton scattering (SCS) leads to a rather complicated difference equation of the type

$$\begin{aligned} &\mathrm{i}\, \mathrm{d}C_l/\mathrm{d}\tau = (\alpha + \mu l) l C_l + \Omega\{[(l+1)(n_- - l)]^{1/2} C_{l+1} + [l(n_- - l + 1)]^{1/2} C_{l-1}\}, \\ &\qquad C_l(0) = \delta_{l,0}, \end{aligned} \tag{1}$$

where l is a positive integer, μ and Ω are known constants and n_- is a fixed, arbitrary positive integer. The function is related to the complex amplitude of the SCS process, i.e. the scattering of light in presence of a stimulating radiation field.

We have already analysed in Bosco *et al* (1984) a particular case of (1), i.e.

$$\begin{aligned} &\mathrm{i}\, \mathrm{d}C_l/\mathrm{d}\tau = \Omega\{[(l+1)(n_- - l)]^{1/2} C_{l+1} + [l(n_- - l + 1)]^{1/2} C_{l-1}\}, \\ &C_l(0) = \delta_{l,0}, \end{aligned} \tag{2}$$

and we have found its solution by means of an operational technique involving the so-called 'simple split three-dimensional Lie algebra'.

We present in this paper a further step in the analysis of (1) by working out the exact solution of the equation

$$\begin{aligned} &\mathrm{i}\, \mathrm{d}C_l/\mathrm{d}\tau = \alpha l C_l + \Omega\{[(l+1)(n_- - l)]^{1/2} C_{l+1} + [l(n_- - l + 1)]^{1/2} C_{l-1}\}, \\ &C_l(0) = \delta_{l,0}, \end{aligned} \tag{3}$$

which is the necessary step for the perturbative solution of (1) in terms of the small parameter μ. In the SCS process the parameter μ is intimately connected to the electron quantum recoil. We point out, however, that the solution of (3) would be interesting on its own for it describes the interaction of a Bloch state with an external field. (Arecchi *et al* 1972).

In several previous papers (Bosco *et al* 1984, Bosco and Dattoli 1983, Ciocci *et al* 1984, Dattoli *et al* 1984) we developed an operational technique to analyse a wide class of differential-difference equations of the Raman–Nath type. In this paper we exploit a generalisation of that method to discuss the solution of (3) known as the Spherical Raman–Nath (SRN) equation. As a preliminary step we perform the following transformations:

$$C_l(x)=(-\mathrm{i})^l\exp(-\mathrm{i}\,\beta l x)M_l(x), \qquad \beta=\alpha/\Omega, \qquad x=\Omega\tau. \tag{4}$$

Therefore (3) can be expressed in the form

$$\begin{aligned} &\mathrm{d}M_l/\mathrm{d}x=-[(l+1)(n_--l)]^{1/2}\exp(-\mathrm{i}\beta x)M_{l+1}+[l(n_--l+1)]^{1/2}\exp(\mathrm{i}\beta x)M_{l-1},\\ &M_l(0)=\mathrm{i}^l\delta_{l,0} \end{aligned} \tag{5}$$

As already remarked in Bosco *et al* (1984) we use angular momentum type operators $J_\pm$ defined as

$$J_+M_l=[(l+1)(n_--l)]^{1/2}M_{l+1}, \qquad J_-M_l=[l(n_--l+1)]^{1/2}M_{l-1}, \tag{6}$$

to write (5) as follows:

$$\mathrm{d}M_l/\mathrm{d}x=[-\exp(-\mathrm{i}\beta x)J_++\exp(\mathrm{i}\beta x)J_-]M_l(x). \tag{7}$$

Using the standard rules of commutation of angular momentum we can define a 'simple split three-dimensional algebra'

$$[J_+,-J_-]=-2J_z, \qquad [J_+,-2J_z]=2J_+, \qquad [-J_-,-2J_z]=2J_- \tag{8}$$

where J_z is the third component of our angular momentum operator and its action on M_l is defined below:

$$J_zM_l=(l-n_-/2)M_l. \tag{9}$$

To find a solution of (7) we can use the Wei–Norman (1963) Lie algebraic approach to the linear differential equations, yielding

$$M_l=\exp[-2h(x)J_z]\exp[g(x)J_+]\exp[-f(x)J_-]M_l(0), \tag{10}$$

where $h(x)$, $g(x)$ and $f(x)$ are three scalar functions obeying the following differential equations

$$\begin{aligned} &\mathrm{d}h(x)/\mathrm{d}x=g(x)\exp[-\mathrm{i}\beta x-2h(x)],\\ &\mathrm{d}g(x)/\mathrm{d}x=-\exp[-\mathrm{i}\beta x+2h(x)]+g^2(x)\exp[\mathrm{i}\beta x-2h(x)],\\ &\mathrm{d}f(x)/\mathrm{d}x=-\exp[\mathrm{i}\beta x-2h(x)], \end{aligned} \tag{11}$$

with initial conditions $h(0)=g(0)=f(0)=0$.

The solution of the system of differential equations (11) is reduced to quadratures if we know the solution of the Riccati equation

$$\begin{aligned} &\mathrm{d}U(x)/\mathrm{d}x+U^2(x)-\mathrm{i}\beta U(x)+1=0,\\ &U(x)=\mathrm{d}h(x)/\mathrm{d}x, \qquad U(0)=0. \end{aligned} \tag{12}$$

After some simple algebra we find

$$\begin{aligned} &h(x)=\mathrm{i}\tfrac{1}{2}\beta x+\ln[\cos(-\tfrac{1}{2}\delta x+\varphi)/\cos\varphi],\\ &g(x)=-[\cos[\tfrac{1}{2}\delta x-\varphi)/\cos\varphi]^2[\tfrac{1}{2}\delta\tan(\tfrac{1}{2}\delta x-\varphi)+\mathrm{i}\tfrac{1}{2}\beta],\\ &f(x)=-[(2\cos^2\varphi)/\delta][\tan(\tfrac{1}{2}\delta x-\varphi)-\mathrm{i}\beta/\delta], \end{aligned} \tag{13}$$

with $\delta=(\beta^2+4)^{1/2}$ and $\tan\varphi=-\beta/\delta$.

$C_l(x)$ is found by expanding the exponents in (11), and using (6) and (9) we obtain

$$C_l(\tau) = \mathrm{i}^l \exp(-\mathrm{i}\alpha l\tau)\binom{n_-}{l}^{1/2} \exp[n_- h(\tau)][f(\tau)]^l$$
$$= (-\mathrm{i})^l\binom{n_-}{l}^{1/2} \exp\left\{\mathrm{i}\alpha l\tau - \mathrm{i}\tan^{-1}\left[\frac{\beta}{\delta}\tan\left(\frac{\delta\Omega}{2}\tau\right)\right](n_- - l)\right\} p^{l/2}(1-p)^{(n-l)/2} \tag{14}$$

where $p(\tau) = (4/\delta^2)\sin^2(\delta\Omega\tau/2)$. It is easy to see that in the limit $\alpha \to 0$ the above result reduces to

$$C_l(\tau) = (-\mathrm{i})^l\binom{n_-}{l}^{1/2}[\tan(\Omega\tau)]^l[\cos(\Omega\tau)]^{n_-} \tag{15}$$

which is the solution of (2) already found in Bosco *et al* (1984) using a slightly different technique.

Equation (14) in the very large n_- limit ($n_- \gg l$ and $\Omega\sqrt{n_-} = \bar{\Omega} =$ constant) gives the solution of the harmonic Raman–Nath equation written in terms of the Poisson–Charlier polynomials (Szëgo 1959) namely

$$C_l \approx (-\mathrm{i})^l \exp\{-\mathrm{i}(n_- - l)\tan^{-1}[(\beta/\delta)\tan(\delta\Omega\tau/2)]\}$$
$$\times \exp(-\mathrm{i}\alpha l\tau)[(N^{l/2}/\sqrt{l!})\exp(-N/2)] \tag{16}$$

where $N = [\bar{\Omega}(\sin\alpha/2)/(\alpha/2)]^2$.

The average value of the spontaneously emitted photons is given by

$$\langle l\rangle = \sum_{l=0}^{n_-} l|C_l(1)|^2 = n_- p(1) = n_-\left(\frac{\sin(\delta\Omega/2)}{\delta/2}\right)^2.$$

This result is easily obtained by noting that the distribution function $|C_l|^2$ is a binomial distribution. Likewise, using (16) we obtain

$$\langle l\rangle_{n_-\gg l} = N = \bar{\Omega}^2\left(\frac{\sin(\alpha/2)}{\alpha/2}\right)^2.$$

In a forthcoming publication we shall make use of the exact solution of (3) as shown in (14) to write a perturbative solution of the generalised Raman–Nath equation.

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